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Canonical Forms of Borel-Measurable Mappings $\Delta: [\omega]^\omega \to \mathbb{R}$

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We show that for every Borel-measurable mapping Δ: $[\omega]^\omega \to \mathbb{R}$ there exists $A \in [\omega]^\omega$ and there exists a continuous mapping Γ: $[A]^\omega \to [A]^{\leqslant \omega}$ with $\Gamma(X) \subseteq X$ such that for all $X, Y \in [A]^\omega$ it follows that $\Delta(X) = \Delta(Y)$ iff $\Gamma(X) = \Gamma(Y)$. In a sense, this is generalization of the Erdös–Rado canonization theorem [*J. London Math. Soc.* **25** (1950), 249–255].

0. PRELIMINARIES

(1) Letters k, l, m, n denote finite ordinals, as usual, $k = \{0, ..., k-1\}$.

(2) ω is the smallest infinite ordinal, the set of nonnegative integers.

(3) Letters A, B, C, X, Y, Z denote infinite subsets of ω. $[A]^\omega$ is the set of all infinite subsets of A.

(4) Letters R, S, T denote finite subsets of ω. $[A]^{<\omega}$ is the set of all finite subsets of A.

(5) $[A]^{\leqslant\omega}$ is the set of all subsets of A, this can be identified with 2^A, the set of mappings τ: $A \to \{0, 1\}$. 2^A, and hence also $[A]^{\leqslant\omega}$, is a topological space, endowed with the product topology (Cantor's discontinuum).

(6) For sets $S, X \in [\omega]^{\leqslant\omega}$ we write $S < X$ iff $\max S < \min X$.

(7) A basis of the topological space $[\omega]^\omega$ is given by all sets $\{S \cup X \mid X \in [\omega]^\omega, S < X\}$, $S \in [\omega]^{<\omega}$.

(8) A subset $\mathscr{B} \subseteq [\omega]^\omega$ is a Borel-set if it belongs to the σ-algebra generated by all open subsets of $[\omega]^\omega$, a mapping Δ: $[A]^\omega \to \mathbb{R}$ is Borel-measurable if for all open subsets $I \subseteq \mathbb{R}$ the preimage $\Delta^{-1}(I)$ is Borel. $\mathbb{R}$ is the set of real numbers.

(9) A subset $A \subset [\omega]^\omega$ is analytic if it is the projection of a Borel-set.

(10) For a brief outline of descriptive set theory, compare, e.g., [3].

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1. Introduction

It is well known that, in general, no Ramsey-type theorem is valid for infinite subsets of ω. Particularly, assuming $V = L$, the axiom of constructibility, there exists a $(\sum_2^1 \cap \prod_2^1)$-mapping Δ: $[\omega]^\omega \to \{0, 1\}$ such that for no $A \in [\omega]^\omega$ the restriction $\Delta \upharpoonright [A]^\omega$ is constant.

However, Galvin and Prikry [2] showed that for every Borel-measurable mapping Δ: $[\omega]^\omega \to \{0, 1\}$ there exists $A \in [\omega]^\omega$ such that $\Delta \upharpoonright [A]^\omega$ is a constant mapping. Silver [5] generalized this, showing that for every analytic subset $\mathscr{A} \subseteq [\omega]^\omega$ there exists $A \in [\omega]^\omega$ such that $[A]^\omega \subset \mathscr{A}$ or $[A]^\omega \cap \mathscr{A} = \varnothing$.

In this paper we investigate arbitrary Borel-measurable mappings Δ: $[\omega]^\omega \to \mathbb{R}$ and show that there always exists $A \in [\omega]^\omega$ such that the structure of $\Delta \upharpoonright [A]^\omega$ can be easily described.

Our main result is

Main Theorem. *Let Δ: $[\omega]^\omega \to \mathbb{R}$ be a Borel-measurable mapping. Then there exists $A \in [\omega]^\omega$ and there exists γ: $[A]^{<\omega} \to \{\mathfrak{s}, \mathfrak{m}\}$ such that the mapping Γ: $[A]^\omega \to [A]^{\leqslant \omega}$ with $\Gamma(X) = \{k \in X \mid \gamma(X \cap k) = \mathfrak{s}\}$ has the following properties*:

(i) *$\Gamma(X) \subseteq X$ for all $X \in [A]^\omega$*,

(ii) *for no $X, Y \in [A]^\omega$ there exists $k \in \Gamma(Y)$ such that $\Gamma(X) = \Gamma(Y) \cap k$, i.e., no $\Gamma(X)$ is a proper initial segment of some $\Gamma(Y)$*,

(iii) *for all $X, Y \in [A]^\omega$ it follows that*

$$\Delta(X) = \Delta(Y) \quad \textit{iff} \quad \Gamma(X) = \Gamma(Y).$$

Corollary 1. *Let Δ: $[\omega]^\omega \to \mathbb{R}$ be Borel-measurable. Then there exists $A \in [\omega]^\omega$ and there exist γ, Γ as in the main theorem satisfying* (i), (ii), (iii) *and*

either $\Gamma(X)$ *is finite for all $X \in [A]^\omega$*,

or $\Gamma(X)$ *is infinite for all $X \in [A]^\omega$*.

If we restrict to Borel-measurable mappings with a countable range, then each $\Gamma(X)$ will be finite. This is slightly stronger than a corresponding result of Pudlak and Rödl [4].

Another corollary is the Erdös–Rado canonization theorem.

Corollary 2 [1]. *Let k be a positive integer and let Δ: $[\omega]^k \to \omega$ be* *an arbitrary mapping. Then there exists* $A \in [\omega]^\omega$ *and there exists* $J \subseteq \{0,..., k-1\}$ *such that for all* $S, T \in [\omega]^k$ *it follows that*

$$\Delta(S) = \Delta(T) \quad \textit{iff} \quad S : J = T : J,$$

where for $s_0 < s_1 < \cdots < s_{k-1}$ *it is* $\{s_0,..., s_{k-1}\} : J = \{s_i \mid i \in J\}$.

2. Proof of Main Theorem

Our first lemma is essentially due to Galvin and Prikry [2]. We learned about this from Simpson (cf. also [6]).

Lemma 1. *Let* Δ: $[\omega]^\omega \to \mathbb{R}$ *be Borel-measurable. Then there exists* $A \in [\omega]^\omega$ *such that the restriction* $\Delta \upharpoonright [A]^\omega$ *is a continuous mapping.*

Proof. Let $(I_j)_{j \in \omega}$ be an enumeration of all open intervals in $\mathbb{R}$ which have rational endpoints. The I_j form a basis for the topology of the reals. Put $A_0 = \omega$ and assume by induction that $A_0,..., A_j$ have been constructed such that for all $i < j$ and all $S \subseteq \{\min A_0,..., \min A_i\}$ either $\Delta(S \cup X) \in I_i$ for all $X \in [A_{i+1}]^\omega$ or $\Delta(S \cup X) \notin I_i$ for all $X \in [A_{i+1}]^\omega$. According to the Galvin–Prikry theorem then there exists $A_{j+1} \in [A_j \setminus \{\min A_j\}]^\omega$ such that for all $S \subseteq \{\min A_0,..., \min A_j\}$ either $\Delta(S \cup X) \in I_j$ for all $X \in [A_{j+1}]^\omega$ or $\Delta(S \cup X) \notin I_j$ for all $X \in [A_{j+1}]^\omega$. By construction then $A = \{\min A_j \mid j < \omega\}$ has the desired properties. ∎

Thus, instead of Borel-measurable mappings, we can restrict to continuous mappings Δ: $[\omega]^\omega \to \mathbb{R}$. For the remainder of this section let Δ: $[\omega]^\omega \to \mathbb{R}$ be an arbitrary but fixed continuous mapping.

Definition. Let $S, T \in [\omega]^{<\omega}$, $A \in [\omega]^\omega$. S and T are *separated* by A iff $\Delta(S \cup X) \neq \Delta(T \cup Y)$ for all $X, Y \in [A]^\omega$ with $S < X$ and $T < Y$. S and T are *mixed* by A iff for no $B \in [A]^\omega$ the sets S and T are separated by B. S and T are *decided* by A iff S and T are separated or mixed by A.

Lemma 2. *Given S, T and A, there exists* $B \in [A]^\omega$ *which decides S and T. If S and T are decided by B, then they are also decided by each* $C \in [B]^\omega$, *and C decides in the same way as B does.*

Proof. Obvious from the definition. ∎

Lemma 3. *Assume that R and S, as well as S and T are mixed by A. Then also R and T are mixed by A.*

Proof. Assume to the contrary that there exists $B \in [A]^\omega$ which separates R and T, say $R \cup S \cup T < B$. Consider the set $\mathscr{A} = \{X \in [B]^\omega \mid \exists Y \in [B]^\omega$ with $\Delta(R \cup Y) = \Delta(S \cup X)\}$. $\mathscr{A}$ is analytic, hence, by Silver's result [5] there exists $C \in [B]^\omega$ with $[C]^\omega \subseteq \mathscr{A}$ or $[C]^\omega \cap \mathscr{A} = \varnothing$. Both cases lead to a contradiction: if $[C]^\omega \subseteq \mathscr{A}$, then S and T are separated by C, if $[C]^\omega \cap \mathscr{A} = \varnothing$, then R and S are separated by C. ∎

DEFINITION. Let $S \in [\omega]^{<\omega}$, $A \in [\omega]^\omega$. S is *strongly separated* by A iff for all $\{m, n\} \subseteq A$, $S < \{m, n\}$, the sets $S \cup \{m\}$ and $S \cup \{n\}$ are separated by A. S is *strongly mixed* by A iff for all $\{m, n\} \subseteq A$, $S < \{m, n\}$, the sets $S \cup \{m\}$ and $S \cup \{n\}$ are mixed by A. S is *strongly decided* iff S is strongly separated or strongly mixed by A.

LEMMA 4. *If S is strongly mixed by A, then S and $S \cup \{a\}$ are mixed by A for every $a \in A$, $S < \{a\}$.*

Proof. Assume to the contrary that S and $S \cup \{a\}$ are not mixed. Then there exists $B \in [A]^\omega$ which separates S and $S \cup \{a\}$, say $S \cup \{a\} < B$. But then $S \cup \{a\}$ and $S \cup \{\min B\}$ are separated by B, contradicting that S is strongly mixed. ∎

LEMMA 5. *Let R be strongly separated by A and let T be arbitrary. Then there exists $B \in [A]^\omega$ such that $\Delta(R \cup X) \neq \Delta(T \cup Y)$ for all $X, Y \in [B]^\omega$ with* $\min X < \min Y$.

Proof. Put $A_0 = A$ and assume by induction that $A_0, \ldots, A_j \in [A]^\omega$ have been constructed such that $R \cup \{\min A_i\}$ and T are decided by A_i for every $0 \leqslant i < j$. Let $A_{j+1} \in [A_j \setminus \{\min A_j\}]^\omega$ such that also $R \cup \{\min A_j\}$ and T are decided by A_{j+1}. Let $B^* = \{\min A_j \mid j < \omega\}$. By construction, for every $m \in B^*$ the sets $R \cup \{m\}$ and T are decided by B^*. B^* does not contain $m \neq n$ such that both $R \cup \{m\}$ and T as well as $R \cup \{n\}$ and T are mixed. Otherwise, by Lemma 3 the sets $R \cup \{m\}$ and $R \cup \{n\}$ would be mixed, contradicting that R is strongly separated. Hence, there exists $B \in [B^*]^\omega$ satisfying the assertion of the lemma. ∎

LEMMA 6. *Let R and T be mixed by A, but assume that both R and T are strongly separated by A. Then there exists $B \in [A]^\omega$ such that for all $b \in B$ with $R \cup T < \{b\}$ the sets $R \cup \{b\}$ and $T \cup \{b\}$ are mixed by B.*

Proof. According to Lemma 5 we can assume that (1) $\Delta(R \cup X) \neq \Delta(T \cup Y)$ for all $X, Y \in [A]^\omega$ with $\min X \neq \min Y$. Put $A_0 = A$ and assume by induction that $A_0, \ldots, A_j$ have been constructed such that for all $0 \leqslant i < j$ the sets $R \cup \{\min A_i\}$ and $T \cup \{\min A_i\}$ are decided by A_j. Let $A_{j+1} \in [A_j \setminus \{\min A_j\}]^\omega$ be such that also $R \cup \{\min A_j\}$ and $T \cup \{\min A_j\}$ are decided by A_{j+1}. Clearly, for every $m \in B^* = \{\min A_j \mid j < \omega\}$ the sets

$R \cup \{m\}$ and $T \cup \{m\}$ are decided by B^*. By the pigeon-hole principle there exists $B \in [B^*]^\omega$ such that either all $R \cup \{m\}$ and $T \cup \{m\}$, $m \in B$, are mixed by B—in which case we are done—or such that all $R \cup \{m\}$ and $T \cup \{m\}$, $m \in B$, are separated by B. This would imply that (2) $\Delta(R \cup X) \neq \Delta(T \cup Y)$ for all $X, Y \in [B]^\omega$ with $\min X = \min Y$. But (1) and (2) together contradict that R and T are mixed. ∎

LEMMA 7. *Given S and A, there exists $B \in [A]^\omega$ which strongly decides S.*

Proof. Put $A_0 = A$ and assume by induction that $A_0, \dots, A_j \in [A]^\omega$ have been constructed such that $S \cup \{\min A_i\}$ and S are decided by A_j for every $0 \leqslant i < j$. Let $A_{j+1} \in [A_j \setminus \{\min A_j\}]^\omega$ be such that also $S \cup \{\min A_j\}$ and S are decided by A_{j+1}. By construction, for every $m \in B^* = \{\min A_j \mid j < \omega\}$ the sets $S \cup \{m\}$ and S are decided by B. Let $B \in [B^*]^\omega$ be such that (a) all $S \cup \{m\}$ and S, $m \in B$, are separated by B, or (b) all $S \cup \{m\}$ and S are mixed by B. If (a) occurs, then S is strongly separated by B. Thus assume that (b) occurs. We claim that for every $m \in B$ and $C \in [B]^\omega$ with $\{m\} < C$, there exists $D \in [C]^\omega$ such that for every $n \in D$ the sets $S \cup \{m\}$ and $S \cup \{n\}$ are mixed by D. Put $C_0 = C$ and assume by induction that $C_0, \dots, C_j \in [C]^\omega$ are such that $S \cup \{b\}$ and $S \cup \{\min C_i\}$ are decided by C_j for all $0 \leqslant i < j$. Let $C_{j+1} \in [C_j \setminus \{\min C_j\}]^\omega$ be such that also $S \cup \{m\}$ and $S \cup \{\min C_j\}$ are decided. Then let $D \in [\{\min C_i \mid i < \omega\}]^\omega$ be such that (c) all $S \cup \{m\}$ and $S \cup \{n\}$, $n \in D$, are mixed by D (in which case we are done with the claim), or such that (d) all $S \cup \{m\}$ and $S \cup \{n\}$, $n \in D$ are separated by D. If (d) occurs, $S \cup \{m\}$ and S would be separated by D, contradicting (b).

Now we finish the proof as follows: Put $B_0 = B$ and assume that $B_0, \dots, B_j$ have been constructed such that for all $0 \leqslant i < j$ and all $m \in \{\min B_{i+1}, \dots, \min B_{j-1}\} \cup B_j$ the sets $S \cup \{\min B_i\}$ and $S \cup \{m\}$ are mixed by B_j. Let $B_{j+1} \in [B_j \setminus \{\min B_j\}]^\omega$ be such that also for all $m \in B_{j+1}$ the sets $S \cup \{\min B_j\}$ and $S \cup \{m\}$ are mixed by B_{j+1}, such an B_{j+1} exists according to the claim. Finally, then $A^* = \{\min B_j \mid j < \omega\}$ strongly mixes S. ∎

LEMMA 8. *There exists an $A \in [\omega]^\omega$ which strongly decides each of its finite subsets.*

Proof. Let $A_0 \in [\omega]^\omega$ be such that A_0 strongly decides the empty set. This exists by Lemma 7. Assume by induction that $A_0, \dots, A_j$ are constructed such that A_j strongly decides every $S \subseteq \{\min A_0, \dots, \min A_{j-1}\}$. According to 2^j applications of Lemma 7 let $A_{j+1} \in [A_j \setminus \{\min A_j\}]^\omega$ be such that also all subsets $S \subseteq \{\min A_0, \dots, \min A_j\}$ are strongly decided by A_{j+1}.

Then $A = \{\min A_j \mid j < \omega\}$ has the desired properties. ∎

By Lemma 8 we can assume that Δ: $[\omega]^{\omega} \to \mathbb{R}$ is not only continuous, but that also ω strongly decides each of its finite subsets.

DEFINITION. The mapping γ: $[\omega]^{<\omega} \to \{\mathscr{s}, \mathscr{m}\}$ is defined by

$$\gamma(S) = \mathscr{s} \quad \text{if} \quad \omega \text{ strongly separates } S,$$
$$= \mathscr{m} \quad \text{if} \quad \omega \text{ strongly mixes } S.$$

The mapping Γ: $[\omega]^{\omega} \to [\omega]^{\leqslant \omega}$ is defined by

$$\Gamma(X) = \{k \in X \mid \gamma(X \cap k) = \mathscr{s}\}.$$

LEMMA 9. *There exists $A \in [\omega]^{\omega}$ such that for no $X, Y \in [A]^{\omega}$ there exists $k \in \Gamma(Y)$ with $\Gamma(X) = \Gamma(Y) \cap k$, in other words, no $\Gamma(X)$ is a proper initial segment of some $\Gamma(Y)$.*

Proof. Consider the set $\mathscr{B} = \{X \in [\omega]^{\omega} \mid \Gamma(X) = \varnothing\}$. $\mathscr{B}$ is a Borel-set, in fact, $\mathscr{B}$ is closed. Hence, by the Galvin–Prikry theorem [2] there exist $A_0 \in [\omega]^{\omega}$ such that $[A_0]^{\omega} \subseteq \mathscr{B}$ or $\mathscr{B} \cap [A_0]^{\omega} = \varnothing$. Assume by induction that $A_0, ..., A_j$ have been constructed in such a way that for all $S \subseteq \{\min A_0, ..., \min A_{j-1}\}$ either $\Gamma(S \cup X) \subseteq S$ for all $X \in [\{\min A_0, ..., \min A_{j-1}\} \cup A_j]^{\omega}$ with $S < X$ or $\Gamma(S \cup X) \nsubseteq S$ for all $X \in [\{\min A_0, ..., \min A_{j-1}\} \cup A_j]^{\omega}$ with $S < X$. Again, invoking Galvin–Prikry's theorem there exists $A_{j+1} \in [A_j \backslash \{\min A_j\}]^{\omega}$ such that the inductive assumption is also satisfied for $A_0, ..., A_{j+1}$. Then $A = \{\min A_j \mid j < \omega\}$ has the desired properties. ∎

DEFINITION. Let $A \in [\omega]^{\omega}$ and let Δ: $[A]^{\omega} \to \mathbb{R}$ be continuous. A is *canonical* iff

(1) A strongly decides each of its finite subsets.

(2) For no $X, Y \in [A]^{\omega}$ there exists $k \in \Gamma(Y)$ with $\Gamma(X) = \Gamma(Y) \cap k$,

(3) if $R, T \in [A]^{<\omega}$, $\gamma(R) = \gamma(T) = \mathscr{s}$, and R and T are mixed by A, then for all $m \in A$ with $R \cup T < \{m\}$ the sets $R \cup \{m\}$ and $T \cup \{m\}$ are mixed by A.

LEMMA 10. *There exists an $A \in [\omega]^{\omega}$ which is canonical.*

Proof. By Lemmas 8 and 9 we can assume that ω itself already satisfies (1) and (2). Let $A_0 = \omega$ and assume that $A_0, ..., A_j$ have been constructed such that A_j satisfies (3) for all $R, T \subseteq \{\min A_0, ..., \min A_{j-1}\}$. According to repeated applications of Lemma 6 there exists $A_{j+1} \in [A_j \backslash \{\min A_j\}]^{\omega}$ such that (3) is also satisfied for all $S, T \subseteq \{\min A_0, ..., \min A_j\}$. By construction then $A = \{\min A_j \mid j < \omega\}$ is canonical. ∎

For the remainder of this section let A be a canonical set. We shall show that A satisfies the assertions of the main theorem.

LEMMA 11. *Let $S_0 \subseteq S_1 \subseteq \cdots$ and $T_0 \subseteq T_1 \subseteq \cdots$ be strictly ascending sequences of finite subsets of A. Assume that for all $i<\omega$ the sets S_i and T_i are mixed by A. Then $\Delta(\bigcup_{i<\omega} S_i) = \Delta(\bigcup_{i<\omega} T_i)$.*

Proof. For every $i<\omega$ let $X_i, Y_i \in [A]^\omega$ be such that $S_i < X_i$, $T_i < Y_i$ and $\Delta(S_i \cup X_i) = \Delta(T_i \cup Y_i)$. These sets exist, as S_i and T_i are mixed by A. Obviously $\lim_{i<\omega} S_i \cup X_i = U_{i<\omega} S_i$ and $\lim_{i<\omega} T_i \cup Y_i = U_{i<\omega} T_i$. Hence the assertion follows from the continuity of Δ. ▮

LEMMA 12. *Let X, $Y \in [A]^\omega$ be such that $\Gamma(X) = \Gamma(Y)$. Then for all $k \in \Gamma(X)$ the sets $X \cap k$ and $Y \cap k$ as well as the sets $X \cap (k+1)$ and $Y \cap (k+1)$ are mixed by A.*

Proof. By definition of γ it follows that $\gamma(X \cap j) = \gamma(Y \cap j) = m$ for all $j < \min X$. Hence, by Lemma 3 and Lemma 4 the sets $\varnothing$ and $X \cap \min \Gamma(X)$ as well as $\varnothing$ and $Y \cap \min \Gamma(X)$ are mixed. Therefore, again, by Lemma 3 the sets $X \cap \min \Gamma(X)$ and $Y \cap \min \Gamma(X)$ are mixed by A. By (3) of the definition of canonical it follows that $X \cap (\min \Gamma(X) + 1)$ and $Y \cap (\min \Gamma(X) + 1)$ are mixed. By induction, using the same arguments repeatedly, the assertion follows. ▮

LEMMA 13. *Let $X, Y \in [A]^\omega$ be such that $\Gamma(X) = \Gamma(Y)$. Then $\Delta(X) = \Delta(Y)$.*

Proof. We distinguish whether $\Gamma(X)$ is finite or infinite. First, let $\Gamma(X)$ be finite, say $k = \max \Gamma(X)$. Then $\gamma(X \cap (k+j)) = \gamma(Y \cap (k+j)) = m$ for all $j>0$. Thus, by Lemmas 3 and 4, the sets $X \cap (k+1)$ and $X \cap (k+j)$ as well as the sets $Y \cap (k+1)$ and $Y \cap (k+j)$ are mixed for each $j>0$. From Lemma 12 we know that $X \cap (k+1)$ and $Y \cap (k+1)$ are mixed. Hence, again by Lemma 4, for every $j>0$ the sets $X \cap (k+j)$ and $Y \cap (k+j)$ are mixed. Clearly, $X = \bigcup_{j>0} X \cap (k+j)$ and $Y = \bigcup_{j>0} Y \cap (k+j)$ are mixed. Clearly, $X = \bigcup_{j>0} X \cap (k+j)$ and $Y = \bigcup_{j>0} Y \cap (k+j)$, so the assertion follows from Lemma 11. Next let $\Gamma(X)$ be infinite. Then $X = \bigcup_{k \in \Gamma(X)} X \cap k$ and $Y = \bigcup_{k \in \Gamma(X)} Y \cap k$ and the assertion follows from Lemmas 11 and 12. ▮

LEMMA 14. *Let X, $Y \in [A]^\omega$ be such that $\Gamma(X) \neq \Gamma(Y)$. Then $\Delta(X) \neq \Delta(Y)$.*

Proof. Let m be maximal such that $\Gamma(X) \cap m = \Gamma(Y) \cap m$ and let k (resp. l) be the minimal elements in $\Gamma(X) \backslash \Gamma(X) \cap m$ (resp. in $\Gamma(Y) \backslash \Gamma(Y) \cap m$), k and l exist according to (2) of the definition of canonical. The proof of Lemma 12 actually shows that $X \cap k$ and $Y \cap l$ are mixed by A. Also, $k \neq l$, by choice of m, say $k < l$. From (3) of the definition of canonical we infer that $(X \cap k) \cup \{l\}$ and $(Y \cap l) \cup \{l\}$ are mixed. As $\gamma(X \cap k) = \mathscr{s}$, the sets $(X \cap k) \cup \{k\}$ and $(X \cap k) \cup \{l\}$ are separated. Hence, by Lemma 3, also $X \cap (k+1) = (X \cap k) \cup \{k\}$ and $Y \cap (l+1) = (Y \cap l) \cap \{l\}$ are separated. Thus $\Delta(X) \neq \Delta(Y)$. ▮

3. Proofs of Corollaries

(ad 1) According to the main theorem, we can assume that Δ: $[A]^{\omega} \to \omega$ admits γ: $[A]^{<\omega} \to \{\mathscr{s}, \mathscr{m}\}$ satisfying (i), (ii), and (iii). Let $\mathscr{B} = \{X \in [A]^{\omega} \mid \Gamma(X)$ is finite$\}$. $\mathscr{B}$ is a Borel-set, in fact, $\mathscr{B}$ is open. By Galvin–Prikry's theorem [2] there exists $B \in [A]^{\omega}$ with $[B]^{\omega} \subseteq \mathscr{B}$ or $[B]^{\omega} \cap \mathscr{B} = \varnothing$. ▮

(ad 2) Given Δ: $[\omega]^k \to \omega$, consider the mapping Δ^*: $[\omega]^{\omega} \to \omega$ with $\Delta^*(X) = \Delta(X \urcorner k)$, where $X \urcorner k$ denotes the set of the k first elements of X. According to the main theorem let $A \in [A]^{\omega}$, γ and Γ satisfy (i), (ii), (iii) with respect to Δ^*. Obviously, $\Gamma(X) \subseteq X \urcorner k$ for every $X \in [A]^k$, thus Γ can be viewed as a mapping Γ: $[A]^k \to [A]^{\leq k}$. According to Ramsey's theorem there exists $B \in [A]^{\omega}$ and there exists $J \subseteq \{0, ..., k-1\}$ such that $\Gamma(S) = S : J$ for all $S \in [B]^k$. ▮

4. Concluding Remarks

(1) For every Borel-measurable mapping Δ: $[\omega]^{\omega} \to \mathscr{M}$, where $\mathscr{M}$ is a metric space, the image of Δ is separable (cf. [6]). Using the same proof as for Lemma 1 then shows that $\Delta \urcorner [A]^{\omega}$ is continuous for some $A \in [\omega]^{\omega}$. Hence, our main theorem remains valid if we replace $\mathbb{R}$ by an arbitrary metric space $\mathscr{M}$.

(2) In fact, even more is true. The *Ellentuck-topology* on $[\omega]^{\omega}$ is the topology whose basic open sets are $\{S \cup X \mid X \in [A]^{\omega}\}$ for $S \in [\omega]^{<\omega}$ and $A \in [\omega]^{\omega}$, cf. [7]. Louveau and Simpson [8] showed that for every mapping Δ: $[\omega]^{\omega} \to \mathscr{M}$, where $\mathscr{M}$ is a metric space, with the property that inverse images of open sets have the property of Baire with respect to the Ellentuck topology, there exists $A \in [\omega]^{\omega}$ such that $\Delta \urcorner [A]^{\omega}$ has a separable image. This implies that Lemma 1, and hence our main theorem, is valid for all such mappings.

(3) It is reasonable to ask, whether our main theorem is valid for Borel-partitions on $[\omega]^{\omega}$. Thereby, we call an equivalence relation $\pi \subseteq [\omega]^{\omega} \times [\omega]^{\omega}$ a Borel-partition iff π is a Borel-set. Clearly, every Borel-measurable mapping Δ: $[\omega]^{\omega} \to \mathbb{R}$ gives rise to a Borel-partition π by

$X \approx Y \pmod{\pi}$ iff $\Delta(X) = \Delta(Y)$, but generally not vice versa. Let us call a Borel-partition π *continuous* iff for all sequences $(X_i)_{i<\omega}$ and $(Y_i)_{i<\omega}$ with $X_i \approx Y_i \pmod{\pi}$ it follows that also $\lim X_i \approx \lim Y_i \pmod{\pi}$. Inspecting our proof shows that the main theorem would hold for Borel-partitions iff for every Borel-partition π on $[\omega]^\omega$ there exists an $A \in [\omega]^\omega$ such that π restricted to $[A]^\omega$ is continuous. We are indebted to Simpson for bringing the following example to our attention:

Define π by $X \approx Y \pmod{\pi}$ if $(X \setminus Y) \cup (Y \setminus X)$ is a finite set. Then π is a Borel partition, but for no $A \in [\omega]^\omega$ the restriction of π to $[A]^\omega$ is continuous. This shows also that it is essential for our main theorem to deal with Borel measurable mappings rather than Borel-partitions.

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